Primary decomposition

Now we state the more general version of primary decomp:

Theorem: Let R be Noetherian, M a f.g. R-module. Any proper submodule $M' \subset M$ is the intersection of finitely many primary submodules:

$$M' = \bigcap_{i=1}^{n} M_i$$

where M_i is P_i primary, for prime ideals $P_1, \dots, P_n$. Moreover:

a.) Every associated prime of M/M' occurs among the P_i.

b.) If the intersection is irredundant (i.e. no M_i can be dropped), then the P_i are precisely the associated primes of M/M'.

c.) If the intersection is minimal (i.e. no intersection w/ fewer terms exists), then each associated prime of M/M' is equal to P_i for exactly one i.

First, we generalize the notion of irreducibility to modules:

Def: A submodule $N \subseteq M$ is <u>irreducible</u> if N is not the intersection of two strictly larger submodules.

If R is Noeth. and M f.g., we can express M as the intersection of finitely many irreducible submodules by ACC (otherwise we have an infinite ascending chain of modules by removing one at a time), called an <u>irreducible decomposition</u>.

<u>Proof of theorem</u>: let $M' = \bigcap_i M_i$ be an irreducible decomposition of M'.

Suppose some M_i is <u>not</u> primary and let P, Q be two distinct associated primes of M/M_i. Then

$$R/P \text{ and } R/Q \text{ inject into } M/M_i.$$

The annihilator of every element of M/P is P and of M/Q is Q, so their images in M/M_i intersect in 0. Thus, 0 is reducible, so taking preimages of the two submodules in M, their intersection is M_i, which is thus reducible.

Thus, each M_i is primary, so this is a primary decomposition.

To prove a.) – c.), we factor out M' and assume $M' = 0$.

a.) Suppose $0 = \bigcap_{i=1}^{n} M_i$ is a primary decomposition.

Then $M = M/\cap M_i \longrightarrow \bigoplus M/M_i$ is an injection

since $m \mapsto 0 \iff m \in \cap M_i \iff m = 0$.

Thus $\operatorname{Ass} M \subseteq \bigcup \operatorname{Ass} M/M_i = \{P_1, \dots, P_n\}$.

b.) Now suppose that for each j, $\bigcap_{i \neq j} M_i \neq 0$.

We want to show that each $P_i \in \operatorname{Ass} M$.

Fix j and set $A = \bigcap_{i \neq j} M$, $B = M_j$ so $A \cap B = 0$.

Then $A = A/A\cap B \underset{\substack{\uparrow \\ \text{Isomorphism} \\ \text{theorem}}}{\cong} A+B/B \subseteq M/B = M/M_j$

M_j is P_j-primary, so M/M_j is P_j-coprimary,
so A is as well (since $\operatorname{Ass} A \neq \emptyset$).

Thus, $\{P_j\} = \operatorname{Ass} A \subseteq \operatorname{Ass} M$.

c.) Now suppose the decomposition is minimal.
If $P_i = P_j$, then $M_i \cap M_j$ is P_i-primary.
the P_i must be distinct. $\square$

Suppose $M' \subseteq M$ and $M' = \bigcap_i M_i$ the minimal primary decomp. w/ M_i P_i-primary. If P_i is minimal over $\mathrm{Ann}\, M/M'$, we can find M_i as follows:

Claim: In the situation described above,

$$M_i = \ker\left(M/M' \longrightarrow (M/M')_{P_i}\right).$$

Pf: Just as before, assume $M' = 0$. Consider the following commutative diagram:

$$\begin{tikzcd} & M_{P_i} \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & (M/M_i)_{P_i} \\ & M/M_i \arrow[ur] & \end{tikzcd}$$

$\ker(M \to M/M_i) = M_i$, so if we can show the two right maps are injective, we're done.

M_i is P_i-primary, so P_i contains all the zero divisors on M/M_i, so

$$M/M_i \to (M/M_i)_{P_i} \text{ is an injection.}$$

For the other map, note that

$$\bigcap_j M_j = 0, \text{ so } \varphi : M \to \bigoplus_j M/M_j \text{ is injective.}$$

Thus, $M_{P_i} \to \bigoplus_j (M/M_j)_{P_i}$ is injective.

For $j \neq i$, $P_j \not\subseteq P_i$ by minimality. Thus, take $u \in P_j \setminus P_i$.

M_j is P_j-primary, so some power of u annihilates M/M_j, so $\left(M/M_j\right)_{P_i} = 0$.

Thus, $M_{P_i} \to \left(M/M_i\right)_{P_i}$ is injective. $\square$

Primary decomposition and localization

Suppose we have a minimal primary decomposition

$$M' = \bigcap_{i=1}^{n} M_i \subseteq M, \quad \{P_i\} \text{ corr. primes.}$$

Let $U \subseteq R$ multiplicative, and reindex the M_i so that $P_1, \dots, P_t$ are the P_i not meeting U.

Claim: $U^{-1}M' = \bigcap_{i=1}^{t} U^{-1}M_i$ is a minimal primary decomposition.

Pf: Again assume $M' = 0$.

First note that $0 = \bigcap_{i=1}^{n} U^{-1}M_i$, since localization commutes w/ finite intersections. That is, suppose $\frac{m_i}{u_i} \in U^{-1}M_i$, $\frac{m_j}{u_j} \in U^{-1}M_j$ s.t.

$$\frac{m_i}{u_i} = \frac{m_j}{u_j} \in U^{-1}M_i \cap U^{-1}M_j$$

Then $\exists\ v \in U$ s.t. $vu_jm_i = vu_im_j \in M_i \cap M_j$

so $\dfrac{m_i}{u_i} = \dfrac{vu_jm_i}{vu_ju_i} \in U^{-1}(M_i \cap M_j)$,

so $\displaystyle\bigcap_{i=1}^{n} U^{-1}M_i = U^{-1}\left(\bigcap_{i=1}^{n} M_i\right) = 0.$

If $U \cap P_i = \emptyset$, then $\operatorname{Ass} U^{-1}M_i = \{P_i(U^{-1}R)\}$, so $U^{-1}M_i$ is $P_i(U^{-1}R)$-primary.

If $U \cap P_i \neq \emptyset$, then $U^{-1}\left(M/M_i\right) = 0 \Rightarrow U^{-1}M_i = U^{-1}M$.

Thus $0 = \displaystyle\bigcap_{i=1}^{t} U^{-1}M_i$, and this is a primary decomp.

Moreover, $U^{-1}P_1, \ldots, U^{-1}P_t$ are exactly the associated primes of $U^{-1}M$, so this decomp is minimal. $\square$

Primary decomposition of principal ideals

We mentioned earlier that in $\mathbb{Z}$, or in a PID more generally, we can decompose any ideal as the intersection of ideals generated by powers of primes. We can generalize this to certain principal ideals in integral domains:

Prop: Let R be a Noetherian integral domain. If $f \in R$ can be written $f = u\prod p_i^{e_i}$ s.t. u is a unit and the p_i's are prime, generating distinct ideals, $e_i > 0$, then $(f) = \bigcap (p_i^{e_i})$ is a minimal primary decomposition of (f).

Pf: First we show $(p_i^{e_i})$ is (p_i)-primary.

Clearly a power of (p_i) annihilates $R/(p_i^{e_i})$.

If $r \in R \setminus (p_i)$ and $rm \in (p_i^{e_i})$, then since p_i is prime, $m \in p_i^{e_i}$. i.e. r is a NZD on $R/(p_i^{e_i})$.
Thus, $(p_i^{e_i})$ is (p_i)-primary.

Clearly $(f) \subseteq \bigcap (p_i^{e_i})$, so we need to show the reverse inclusion. We'll prove by induction.

Let $g = u\prod_{i\neq 1} p_i^{e_i}$. Assume that $(g) = \bigcap_{i \neq 1} (p_i^{e_i})$.

We need to show $(g) \cap (p_1^{e_1}) \subseteq (f)$.

Let $r \in (g) \cap (p_1^{e_1})$. Then $r = gq \in (p_1^{e_1})$. $p_1 \nmid g$, so $p_1 \mid q$. Repeating, we get $p_1^{e_1} \mid q \Rightarrow r \in (f)$. $\square$

Of course, not all elements can be written as the product of primes. (We will see an example on the HW of a principal ideal whose associated primes are not principal.) However, in a UFD this always works.